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Transient Blocking Probability in a Loss System with Arbitrary Initial Conditions

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We extend Riordan's work on the transient blocking probability in an $M/M/N$ loss system where he assumes that all servers are initially free. We consider the case when the initial probability distribution is arbitrary and derive an exact expression for the corresponding transient blocking probability.

I. INTRODUCTION

In his book, Riordan [3] considers an $M/M/N$ loss system and derives its transient blocking probability, that is the probability that all N servers are busy. However, he assumes that all servers are initially free, that is with the initial probability distribution vector

$$\begin{aligned}\mathbf{p(0)} &= [p_0(0)\, p_1(0) \cdots p_N(0)]^T \\ &= [1 \quad 0 \quad 0 \cdots 0]^T,\end{aligned}$$

where $p_k(t)$ denotes the probability that k out of N servers are busy at time t. This assumption is a special case and as such the expression derived is not general.

We generalize the expression for the transient blocking probability to take into account any arbitrary initial probability distribution, $\mathbf{p(0)}$. Such as expression could be useful in approximating the blocking probability of an $M(t)/M/N$ loss system, which is an $M/M/N$ loss system with a time-dependent arival rate [4, 5].

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II. Transient Blocking Probability

An $M/M/N$ loss system is described by the set of differential equations,

$$\begin{aligned}
\dot{p}_0(t) &= -ap_0(t) + p_1(t) \\
\dot{p}_k(t) &= ap_{k-1}(t) - (a+k)\,p_k(t) + (k+1)\,p_{k+1}(t), \qquad k = 1, 2, ..., N-1 \quad (1) \\
\dot{p}_N(t) &= ap_{N-1}(t) - Np_N(t),
\end{aligned}$$

where a is the offered traffic, and the mean service rate, μ, is normalized to one so that the time t is measured in units of the mean service time. We assume that the initial probability distribution is arbitrary, that is,

$$\mathbf{p(0)} = \mathbf{p}_0.$$

In matrix form, (1) is

$$\dot{\mathbf{p}}(\mathbf{t}) = \mathbf{Ap(t)}, \qquad \mathbf{p(0)} = \mathbf{p}_0, \tag{2}$$

where

$$\mathbf{A} = \begin{pmatrix} -a & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ a & -a-1 & 2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & a & -a-2 & 3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a & -a-(N-1) & N \\ 0 & 0 & 0 & 0 & \cdots & 0 & a & -N \end{pmatrix}.$$

Defining

$$\pi_k(s) = \int_0^\infty e^{-su} p_k(u)\, du$$

as the Laplace transforms of $p_k(t)$, (2) becomes

$$\mathbf{D}\pi = \mathbf{p}_0, \tag{3}$$

where

$$\mathbf{D} = \begin{pmatrix} a+s & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -a & a+1+s & -2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -a & a+2+s & -3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -a & a+(N-1)+s & -N \\ 0 & 0 & 0 & 0 & \cdots & 0 & -a & N+s \end{pmatrix}$$

and

$$\pi = [\pi_0(s)\ \pi_1(s) \cdots \pi_N(s)]^T.$$

We are interested in $p_N(t)$, the transient blocking probability, assuming any initial condition. Using Cramer's rule on (3), we obtain

$$\pi_N(s) = |\mathbf{D}_N|/|\mathbf{D}| \tag{4}$$

where $\mathbf{D}_N$ is the matrix formed by replacing the final column of $\mathbf{D}$ with the vector $\mathbf{p}_0$. Following Riordan [3], we define

$$\begin{aligned} d_0(s) &= 1 \\ d_1(s) &= a + s \\ d_{k+1}(s) &= (a + k + s)\, d_k(s) - akd_{k-1}(s) \qquad \text{for} \quad k = 1, 2, ..., N-1. \end{aligned} \tag{5}$$

Then, using the final row to determine the determinant, we obtain

$$\begin{aligned} \mathbf{D}_1 &= p_1 d_1 + ap_0 \\ \mathbf{D}_2 &= p_2 d_2 + a\,|\mathbf{D}_1| \\ &\ \vdots \\ \mathbf{D}_N &= p_N d_N + a\,|\mathbf{D}_{N-1}| \\ &= p_N d_N + ap_{N-1} d_{N-1} + a^2 p_{N-2} d_{N-2} + \cdots + a^N p_0 d_0, \end{aligned} \tag{6}$$

where $p_k = p_k(0)$, $k = 0, 1, ..., N$.
Hence

$$\pi_N(s) = \frac{1}{|\mathbf{D}|} \sum_{k=0}^{N} a^{N-k} p_k(0)\, d_k(s). \tag{7}$$

Riordan gives

$$|\mathbf{D}| = sd_N(s+1). \tag{8}$$

Hence

$$\pi_N(s) = \sum_{k=0}^{N} a^{N-k} p_k(0)(d_k(s)/sd_N(s+1)). \tag{9}$$

Using partial fraction expansion [1] on (9), we obtain, writing $p_k(0) = \beta_k$,

$$\pi_N(s) = \sum_{k=0}^{N} a^{N-k}\beta_k d_k(0)/sd_N(1) + \sum_{k=0}^{N} a^{N-k}\beta_k \sum_{j=1}^{N} d_k(s_j)/[s_j(s-s_j)\, d'_N(s_j+1)], \tag{10}$$

where s_j are the N distinct, negative roots [2] of

$$d_N(s+1) = 0. \tag{11}$$

They can be numerically computed using a method outlined by Machihara [2].

Noting that

$$d_k(0) = a^k$$

$$d_N(1) = \sum_{k=0}^{N} N!\, a^{N-k}/(N-k)!$$

$$\sum_{k=0}^{N} \beta_k = 1,$$

Eq. (10) can be simplified to

$$\pi_N(s) = \frac{a^N/N!}{s\sum_{k=0}^{N} a^k/k!} + \sum_{k=0}^{N}\sum_{j=1}^{N} a^{N-k}\beta_k d_k(s_j)/[s_j(s-s_j)\, d'_N(s_j+1)]. \tag{12}$$

Inverting (12) gives

$$p_N(t) = B(N, a) + \sum_{k=0}^{N}\sum_{j=1}^{N} a^{N-k}\beta_k \exp(s_j t)\, d_k(s_j)/[s_j d'_N(s_j+1)], \tag{13}$$

where

$$B(N, a) = \frac{a^N/N!}{\sum_{k=0}^{N} a^k/k!}$$

is the Erlang-B formula. Note that, since $s_j < 0$,

$$\lim_{t\to\infty} p_N(t) = B(N, a)$$

as should be the case.

The above expression (13) gives the exact transient blocking probability for an $M/M/N$ loss system assuming any initial condition.

III. An Example

As an illustration, consider an $M/M/2$ loss system. In this case we have $N=2$, and

$$\begin{aligned} &d_2(s+1)=s^2+s(2a+3)+a^2+2a+2 \\ &\Rightarrow s_1, s_2=-(2a+3\pm\sqrt{1+4a})/2 \\ &d_2'(s+1)=2(s+a)+3. \end{aligned}$$

Therefore

$$\begin{aligned} p_2(t)=B(2,a)&+\frac{a^2+s_2(\alpha\beta_1+s_2\beta_2+2\alpha\beta_2+\beta_2)}{s_1(2s_1+2a+3)}\exp(s_1 t) \\ &+\frac{a^2+s_1(a\beta_1+s_1\beta_2+2a\beta_2+\beta_2)}{s_2(2s_2+2a+3)}\exp(s_2 t). \end{aligned} \tag{14}$$

Note the symmetry in s_1 and s_2. For a more complex systems, s_j would have to be computed numerically [2].

IV. Conclusion

We present an exact expression for the transient blocking probability in the $M/M/N$ loss system, assuming any initial condition. This is a generalization of Riordan's result and could find application in a method to approximate loss systems with nonhomogeneous Poisson arrivals. Such loss systems have been proposed for modelling real-life telecommunications systems, where the time-dependent offered traffic can be approximated by a sequence of step functions of duration Δt [4, 5]. However, Δt is normally too brief for transience to disappear. Hence we need to take into account this transience and the results outlined in this paper give us the transient blocking probability. This measure could then be used to calculate the transient mean of lost traffic, say, from an actual telephone exchange, due to congestion, during Δt. This transient mean, in turn, could be used to approximate actual lost traffic during a longer period of time. This is a subject of further research.

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